Question 1 (15 Marks) **Marks**

(a) Draw a neat sketch of $xy = 8$, **clearly** indicating on the sketch, the coordinates of the foci, vertices, and the equations of the directrices. **4**

(b) A raindrop falls so that its velocity v m/s at time t seconds is given by

$$\frac{dv}{dt} = \frac{1}{3}(3g - 2v)$$

where g is the acceleration due to gravity.

(i) Show that $v = \frac{3g}{2}\left(1 - e^{-\frac{2}{3}t}\right)$. **3**

(ii) Find the limiting velocity of the raindrop in terms of g. **1**

(iii) Find the time when the velocity reaches $\frac{1}{2}g$ m/s. **2**

(c) The rate of increase of the population, $P(t)$, of a particular bird species at time t years is given by the equation:

$$\frac{dP}{dt} = kP(Q - P)$$

where k and Q are positive constants and $P(0) < Q$.

(i) Verify that the expression $P(t) = \dfrac{QC}{C + e^{-kQt}}$, where C is a constant, is a solution of the equation. **3**

(ii) Describe the behaviour of P as $t \to \infty$. **1**

(iii) Describe what happens to the rate of increase of the population as $t \to \infty$. **1**

Question 2 (15 Marks) **(Start a new page)**

(a) A particle of unit mass is projected vertically upwards form the ground with initial speed u m/s. If air resistance at any time t seconds is proportional to the velocity at that instant, and assuming air resistance is $-kv$,

(i) Prove that if the highest point is reached by the particle in time T seconds then **4**

$$kT = \log\left(1 + \frac{ku}{g}\right)$$

where g is the acceleration due to gravity.

(ii) If the highest point reached is at a height h metres above the ground, prove that $hk = u - gT$. **5**

Question 2 continues on the next page

Question 2 continued **Marks**

(b) The normal at a variable point $P\left(2p, \frac{2}{p}\right)$ on $xy = 4$, given by $y = p^2x - 2p^3 + \frac{2}{p}$, meets the x – axis at Q.

(i) Find the coordinates of Q. **1**
(ii) Find the coordinates of the midpoint, M, of PQ. **2**
(iii) Hence, find the locus of M. **3**

Question 3 (15 Marks)

(a) (i) Prove that the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab square units. **3**

(ii) Hence, by the **method of cylindrical shells**, find the volume of the solid formed when the area is rotated through 1 complete revolution about the line $y = b$. **3**

(b) The area enclosed by the graph of the function $y = e^{2x}$, the y – axis and the horizontal line $y = e^2$ is rotated about the y – axis.

(i) Show that the volume is given by $\Delta V = \sum_0^1 2\pi x(e^2 - e^{2x})\Delta x$. **2**

(ii) Hence, find the exact volume of the solid of revolution formed. **3**

(c) If the gradients of the tangents drawn to the curves $xy = c^2$ and $y^2 = 4ax$ at the point of intersection are m and M respectively. Prove that $m = -2M$. **4**

Question 4 (15 Marks) **(Start a new page)** **Marks**

(c) A particle moves in a straight line so that it's acceleration is inversely proportional to the square of its distance from a point O in the line and is directed towards O. It starts from rest at a distance a units from O.

(i) What is its velocity when it first reaches a distance, $\frac{a}{2}$ units, from O? **3**

(ii) Show that the time taken to first reach this distance in part (i) is given by **3**

$$t = \frac{(\pi + 2)a^{\frac{3}{2}}}{4\sqrt{2k}}, \text{ where } k \text{ is a constant,}$$

given that $\frac{d}{dx}\left[\sqrt{x(a-x)} + \frac{a}{2}\sin^{-1}\left(\frac{a-2x}{a}\right)\right] = -\sqrt{\frac{x}{a-x}}$

Question 4 is on the next page.

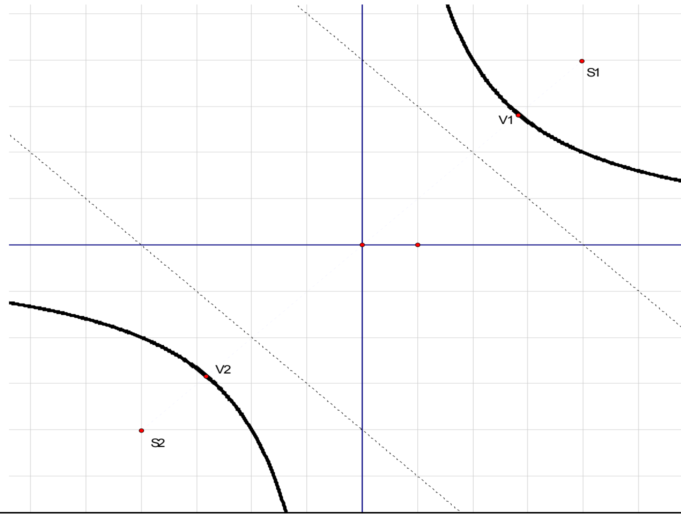
Question 4 continued **Marks**

(b) A is the area of the region R bounded by the upper branch of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, the x – axis and the lines $x = \pm\, a$.

(i) Show that $A = \frac{Lb}{2}\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$ square units, where L the length of the base of R is $2a$ units. **5**

(ii) S is the solid whose base is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Cross sections perpendicular to the base and to the minor axis, are plane figures similar to region R where the line of intersection of the planes is the base length of R. Find the volume of S. **4**

~ END OF TEST ~

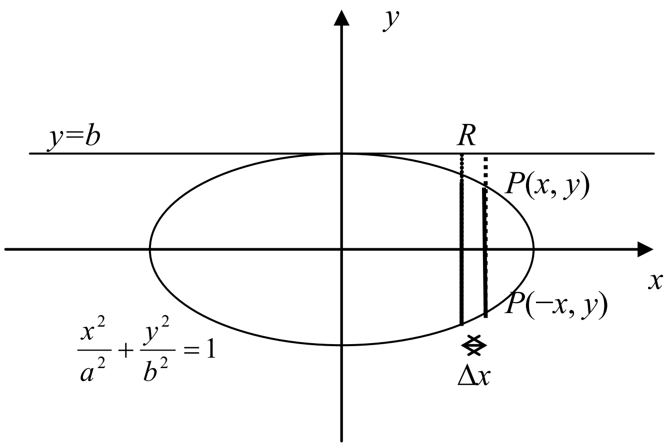
Solutions to Questions	Marking Scheme	Comments
Question 1		
(a) 	• Foci $(\pm 4, \pm 4)$ ➔ **1 mk** • Vertices $\left(\pm 2\sqrt{2}, \pm 2\sqrt{2}\right)$ ➔ **1 mk** • Eqn. of directrices $x + y = \pm 4$ ➔ **1mk** • Shape of graph ➔ **1mk**	Students must **clearly** show the coordinates of the foci and vertices & eqn. of directrices to obtain full marks. **½ mark off is scale is wrong/ no scale given.**
(b) (i) $\frac{dv}{dt} = \frac{1}{3}(3g - 2v)$ $\therefore \int_0^v \frac{dv}{3g - 2v} = \frac{1}{3}\int_0^t dt$ $\therefore -\frac{1}{2}\ln\lvert 3g - 2v\rvert_0^v = \frac{1}{3}[t]_0^t$ $\therefore -\frac{1}{2}\ln\left\lvert\frac{3g - 2v}{3g}\right\rvert = \frac{1}{3}t$ or $\frac{1}{2}\ln\left\lvert\frac{2v - 3g}{3g}\right\rvert = \frac{1}{3}t$ $\therefore \ln\left\lvert\frac{3g - 2v}{3g}\right\rvert = -\frac{2}{3}t$ $\therefore \frac{3g - 2v}{3g} = e^{-\frac{2t}{3}}$ $\therefore v = \frac{3g}{2}\left(1 - e^{-\frac{2t}{3}}\right)$	Correct integral ➔ **1** mk Correct integration ➔ ½ mk Substitution & simplify➔ **½ mk** Taking e to both sides ➔ **1 mk**	Variation to integral can be: $t = \int \frac{3dv}{3g - 2v}$ When $t = 0$, $v = 0$ ➔ $c = -\frac{3}{2}\ln(3g)$

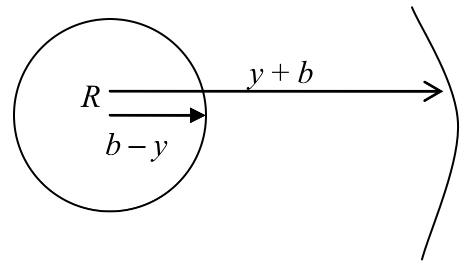
1 (b) (ii) as $t \rightarrow \infty$ $v \rightarrow \dfrac{3g}{2}$	Correct answer ➔ **1 mk**	
1 (b) (iii) when $v = \dfrac{g}{2}$ $\therefore \dfrac{1}{3} = 1 - e^{-\frac{2t}{3}}$ $\therefore -\dfrac{2}{3}t = \ln\left(\dfrac{2}{3}\right)$ $\therefore t = -\dfrac{3}{2}\ln\left(\dfrac{2}{3}\right)$ or $t = \dfrac{3(\ln 3 - \ln 2)}{2}$	Correct substitution & simplification ➔ **1 mk** Taking logs of both sides & simplification ➔ **1 mk**	
1 (c) (i) $P = \dfrac{QC}{C + e^{-kQt}}$ $\therefore \dfrac{dP(t)}{dt} = \dfrac{Q^2Cke^{-kQt}}{\left(C + e^{-kQt}\right)^2}$ $= \dfrac{kQC}{C + e^{-kQt}} \times \dfrac{Qe^{-kQt}}{C + e^{-kQt}}$ $= kP\,(Q - P)$ As $(Q - P) = Q - \dfrac{QC}{C + e^{-kQt}}$ $= \dfrac{QC + Qe^{-kQt} - QC}{C + e^{-kQt}}$ $= \dfrac{Qe^{-kQt}}{C + e^{-kQt}}$	Correct differential ➔ **1 mk** Simplification ➔ **1 mk** Showing $(Q - P) = \dfrac{Qe^{-kQt}}{C + e^{-kQt}}$ ➔ **1 mk**	
1 (c) (ii) as $t \rightarrow \infty$, $P \rightarrow Q$ as $e^{-kQt} \rightarrow 0$	Correct answer with some explaination ➔ **1 mk**	
1 (c) (iii) as $t \rightarrow \infty$, $\dfrac{dP(t)}{dt} \rightarrow 0$ as $P \rightarrow Q$	Correct answer with some explaination ➔ **1 mk**	

Question 2		
(a) (i) $m\ddot{x} = m(-g) - mkv$ (upwards) $\therefore\ \ddot{x} = -g - kv$ i.e. $\frac{dv}{dt} = -g - kv$ $\therefore\ \frac{1}{k}\int_u^0 \frac{k\ dv}{-(g+kv)} = \int_0^T dt$ $\therefore\ -\frac{1}{k}\left[\ln\lvert g+kv\rvert\right]_u^0 = T$ $\therefore\ -\ln\left\lvert\frac{g}{g+ku}\right\rvert = kT$ $\therefore\ \ln\left\lvert\frac{g+ku}{g}\right\rvert = kT$ ➔ $kT = \ln\left\lvert 1+\frac{ku}{g}\right\rvert$	Correct equation ➔ **1 mk** Correct integral ➔ **1 mk** Correct integration ➔ **1 mk** Correct simplification ➔ **1 mk**	
2 (a) (ii) Highest point reached is when $v = 0$ $\therefore\ \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -g - kv$ ➔ $v\frac{dv}{dx} = -g - kv$ $\therefore -\int_u^0 \frac{v\ dv}{g+kv} = \int_0^h dx$ $\therefore\ -\int_u^0 \frac{1}{k} - \frac{g\ dv}{k(g+kv)} = \int_0^h dx$ $\therefore\ -\int_u^0 1 - \frac{g\ dv}{(g+kv)} = kh$ $\therefore\ -\left[v - \frac{g}{k}\ln(g+kv)\right]_u^0 = kh$ $\therefore\ -\left[-u + \frac{g}{k}\ln\left(\frac{g+ku}{g}\right)\right] = hk$ $\therefore\ hk = u - gT$ as $kT = \ln\left\lvert 1+\frac{ku}{g}\right\rvert$ from part(i)	$v\frac{dv}{dh} = -g - kv$ ➔ **1 mk** Correct integral ➔ **1 mk** $\frac{vdv}{g+kv} = \frac{1}{k} - \frac{gdv}{k(g+kv)}$ ➔ **1 mk** Correct integration ➔ **1 mk** Correct simplification & connecting answer from (i) ➔ **1 mk**	

2 (b) $xy = 4$ Eqn. of normal is $y = p^2x - 2p^3 + \frac{2}{p}$ **(i)** $\therefore$ coordinates of Q are $\left(2p - \frac{2}{p^3}, 0\right)$	Correct coordinates ➔ **1 mk**	
(ii) $M = \left[\left(\frac{2p + 2p - \frac{2}{p^3}}{2}\right), \left(\frac{\frac{2}{p}}{2}\right)\right]$ $M = \left[\left(2p - \frac{1}{p^3}\right), \frac{1}{p}\right]; p \neq 0$	Correct midpoint formula ➔ **1 mk** Correct simplification & restriction for p ➔ **1 mk**	Alternatively can award 1 mk each for x and y coordinate of midpoint.
(iii) $x = 2p - \frac{1}{p^3}; y = \frac{1}{p}$ $\therefore p = \frac{1}{y}$; $y \neq 0$ $\therefore$ $\mathbf{x = \frac{2}{y} - y^3}$ **is the locus of** $\mathbf{M}$ **;** $\mathbf{y \neq 0}$	➔ **1 mk** ➔ **1 mk** equation ➔ **1 mk** restriction $y \neq 0$	
Question 3		
(a) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ➔ $y = \frac{b}{a}\sqrt{a^2 - x^2}$ $\therefore$ Area $= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$ $= \frac{4b}{a} \times \frac{1}{4}\pi a^2$ since $\int_0^a \sqrt{a^2 - x^2}\,dx$ is a quadrant of a circle, centre O radius a units. $\therefore$ Area $= \pi ab$ sq. units.	➔ **1 mk** For $\frac{1}{4}\pi a^2$ ➔ **1 mk** ➔ **1 mk** explanation of using $\frac{1}{4}\pi a^2$	

3 (a) (ii)





A slice taken through the ellipse perpendicular to the x – axis is the annulus with inner radius $(b - y)$ and outer radius $(b + y)$.

$\therefore$ Area of cross- section of slice

$= \pi[(b + y)^2 - (b - y)^2] = 4\pi by$

Showing area of cross section is $4\pi ay$ ➔ **1 mk**

$\therefore$ Volume of slice $\Delta V = 4\pi by\, \Delta x$

$\therefore$ Volume of solid

$$V = 4\pi b \int_{-a}^{a} y\,dx$$

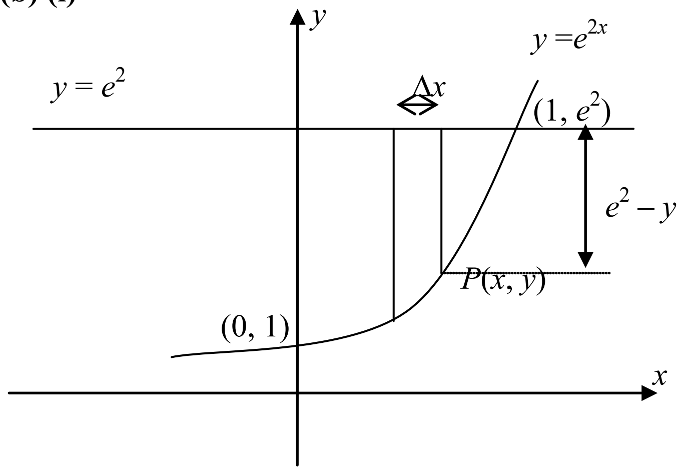
$$= 8\pi b \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

Correct integral ➔ **1 mk**

$$= \frac{8\pi b^2}{a} \times \frac{1}{4}\pi a^2 \text{ from part (i)}$$

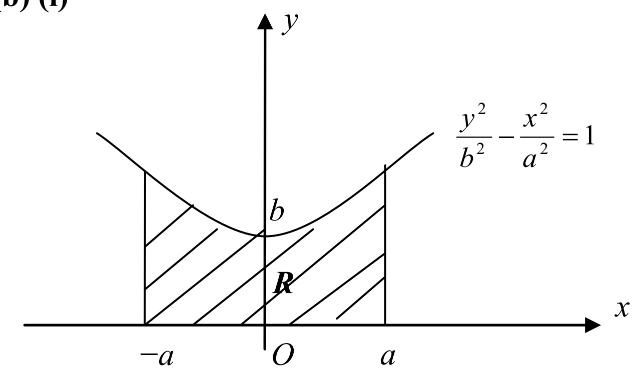
$= 2\pi^2 ab^2$ cubic units.

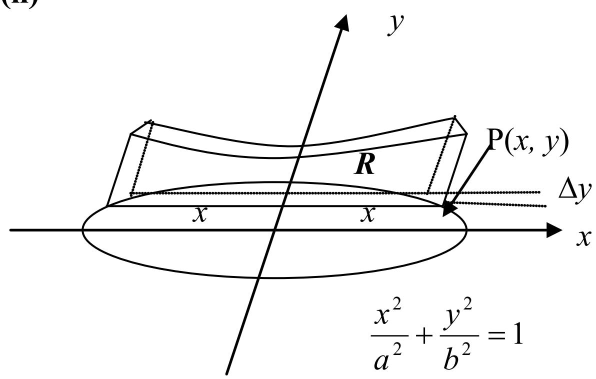
Correct Answer ➔ **1 mk**

3 (b) (i) 		
Area of rectangular slice; $A(x) = 2\pi x(e^2 - e^{2x})$	➔ **1 mk**	
$\therefore \Delta V = 2\pi x(e^2 - e^{2x})\Delta x$		
$\therefore$ Vol = $\lim_{\Delta x \to 0} \sum_{0}^{1} 2\pi x(e^2 - e^{2x})\Delta x$	➔ **1 mk**	
3 (b) (ii) $\therefore$ Vol $= \int_0^1 2\pi x\left(e^2 - e^{2x}\right)dx$		
$= \left[2\pi e^2 \cdot \frac{1}{2}x^2\right]_0^1 - \pi\int_0^1 2xe^{2x}dx$	➔ **1 mk**	
$= \pi e^2 - \pi\left[xe^{2x}\right]_0^1 + \pi\int_0^1 e^{2x}dx$	➔ **1 mk**	
$= \pi e^2 - \pi e^2 + \frac{1}{2}\pi\left[e^{2x}\right]_0^1 = \frac{1}{2}\pi\left(e^2 - 1\right)$	➔ **1 mk**	

3 (c) Substituting $y^2 = 4ax$ into $xy = c^2$ we get $y^3 = 4ac^2 = 2a^3$ as $2c^2 = a^2$. Let P be the point of intersection where $\therefore\ y = a\sqrt[3]{2}$ and $x = \dfrac{a\left(\sqrt[3]{4}\right)}{4}$ By differentiating $xy = c^2$ we get $\dfrac{dy}{dx} = -\dfrac{y}{x}$ $\therefore$ the gradient of the hyperbola $xy = c^2$ at P is $m = \dfrac{-a\left(\sqrt[3]{2}\right)}{\left(\frac{a}{4}\right)\left(\sqrt[3]{4}\right)} = \dfrac{-4}{\sqrt[3]{2}} \rightarrow \boxed{A}$ By differentiating $y^2 = 4ax$ we get $\dfrac{dy}{dx} = \dfrac{2a}{y}$ $\therefore$ gradient of $y^2 = 4ax$ is given by $M = \dfrac{2a}{a\left(\sqrt[3]{2}\right)} = \dfrac{2}{\sqrt[3]{2}} \rightarrow \boxed{B}$ $\therefore$ A ÷ B ➔ $m = -2M$ as required.	➔ **1mk** ➔ **½ mk** ➔ **1 mk** ➔ **½ mk** ➔ **1 mk**	

4(a) (i) $\ddot{x} = -\frac{k}{x^2}$ $\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{k}{x^2}$ $\therefore \frac{1}{2}v^2 = -\int \frac{k}{x^2}dx = \frac{k}{x} + c$ Now $v = o$ when $x = a$ $\therefore c = -\frac{k}{x}$ $\therefore v^2 = 2k\left(\frac{1}{x} - \frac{1}{a}\right)$ Now $0 < x < a$ but motion is moving towards the origin for $t > 0$. $\therefore v = -\sqrt{2k\left(\frac{1}{x} - \frac{1}{a}\right)}$ For $x = ½ a$, $v = -\sqrt{\frac{2k}{a}}$	➔ **½ mk** ➔ **½ mk** ➔ **1 mk** Correct answer➔ **1 mk**	
4 (a) (ii) $\therefore \frac{1}{v} = \frac{dt}{dx} = -\frac{1}{\sqrt{2k\left(\frac{a-x}{ax}\right)}} = -\frac{1}{\sqrt{\frac{2k}{a}\left(\frac{a-x}{x}\right)}}$ $= -\sqrt{\frac{a}{2k}} \cdot \sqrt{\frac{x}{a-x}} \quad = \sqrt{\frac{a}{2k}} \cdot \int_{\frac{a}{2}}^{a} \sqrt{\frac{x}{a-x}}$ $= -\sqrt{\frac{a}{2k}}\left[\sqrt{x(a-x)} + \frac{1}{2}a\sin^{-1}\left(\frac{a-2x}{a}\right)\right]_{\frac{a}{2}}^{a}$ $= -\sqrt{\frac{a}{2k}}\left[\frac{1}{2}a\sin^{-1}(-1) - \frac{a}{2} - \frac{1}{2}a\sin(0)\right]$ $\therefore t = \frac{(\pi + 2)a^{\frac{3}{2}}}{4\sqrt{2k}}$	Correct $\frac{1}{v}$ equation➔ **1 mk** $\sqrt{\frac{a}{2k}} \cdot \int_{\frac{a}{2}}^{a} \sqrt{\frac{x}{a-x}}$ ➔ **1 mk** Correct substitution ➔ **1mk**	

4 (b) (i) 		
$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ can be written as $y=\pm\frac{b}{a}\sqrt{a^2+x^2}$ $\therefore$ Area of region R is given by Area = $\frac{2b}{a}\int_0^a\sqrt{a^2+x^2}$ Let $x = a\tan\theta$. $\therefore$ at $x = a$, $\tan\theta = 1$ $\therefore$ $\theta=\frac{\pi}{4}$ At $x = 0$, $\tan\theta = 0$ so $\theta = 0$. Also $dx = a\sec^2\theta\, d\theta$		
$\therefore$ Area = $A = \frac{2b}{a}\int_0^{\frac{\pi}{4}}\sqrt{a^2+a^2\tan^2\theta}\cdot\sec^2\theta\, d\theta$	➔ **1 mk**	
$= 2ab\int_0^{\frac{\pi}{4}}\sqrt{1+\tan^2\theta}\cdot\sec^2\theta\, d\theta$		
$=2ab\int_0^{\frac{\pi}{4}}\sec^3\theta\, d\theta$ where $I=\int\sec^3\theta\, d\theta$	➔ **1 mk**	

4 (b) (i) Continued Now $I = \int \sec\theta(\sec^2\theta)d\theta = \int \sec\theta(1+\tan^2\theta)d\theta$ $\therefore \quad I = \int \sec\theta\, d\theta + \int (\sec\theta\tan\theta)\tan\theta\, d\theta$ $= \int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}d\theta + \int \tan\theta\cdot\frac{d}{d\theta}(\sec\theta)d\theta$ $= \ln(\sec\theta + \tan\theta) + \tan\theta\sec\theta - I$ $\therefore I = \frac{1}{2}\ln(\sec\theta + \tan\theta) + \frac{1}{2}\tan\theta\sec\theta$ So Area $= ab[\ln(\sec\theta + \tan\theta) + \tan\theta\sec\theta]_0^{\frac{\pi}{4}}$ $= ab\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$ Since $L = 2a$ area generated is $A = \frac{Lb}{2}\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$ unit^2	→ **1 mk** → **1mk** → **1mk**	
4 (b) (ii)  Consider the cross section at $P(x, y)$ on the ellipse of thickness Δy (See diagram). The area of this cross section from (i) is A square units.		

Note $L = 2a = 2x$ so $a = x$. $\therefore$ Area $= A = xb\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$ But $x = \dfrac{a}{b}\sqrt{b^2 - y^2}$ and let $K = \left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$ $\therefore A(y) = \dfrac{LK}{2}\sqrt{b^2 - y^2}$ Now volume of slice, $\Delta V = A(y)\ \Delta y$ $\therefore\ \Delta V = \dfrac{LK}{2}\sqrt{b^2 - y^2}\ \Delta y$ $\therefore$ volume of sum of slices, $V = \dfrac{LK}{2}\lim\limits_{\Delta y \to 0}\sum\limits_{-b}^{b}\sqrt{b^2 - y^2}\,\Delta y$ $= \dfrac{LK}{2}\int_{-b}^{b}\sqrt{b^2 - y^2}\,dy$ $= \dfrac{LK}{2}\cdot\dfrac{1}{2}\pi b^2$ (Note: this integral gives area of semi circle radius b) $\therefore$ Volume of $S = \dfrac{\pi L b^2}{4}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$ in terms of L and b or $= \dfrac{\pi a b^2}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$ in terms of a and b.	$A(y)$ ➔ **1 mk** ➔ **1 mk** ➔ **1 mk** ➔ **1mk**	

~ End of Test ~